Connecting Tribimaximal and Bitrimaximal Mixings*

Carlos Alvarado
Vassar College

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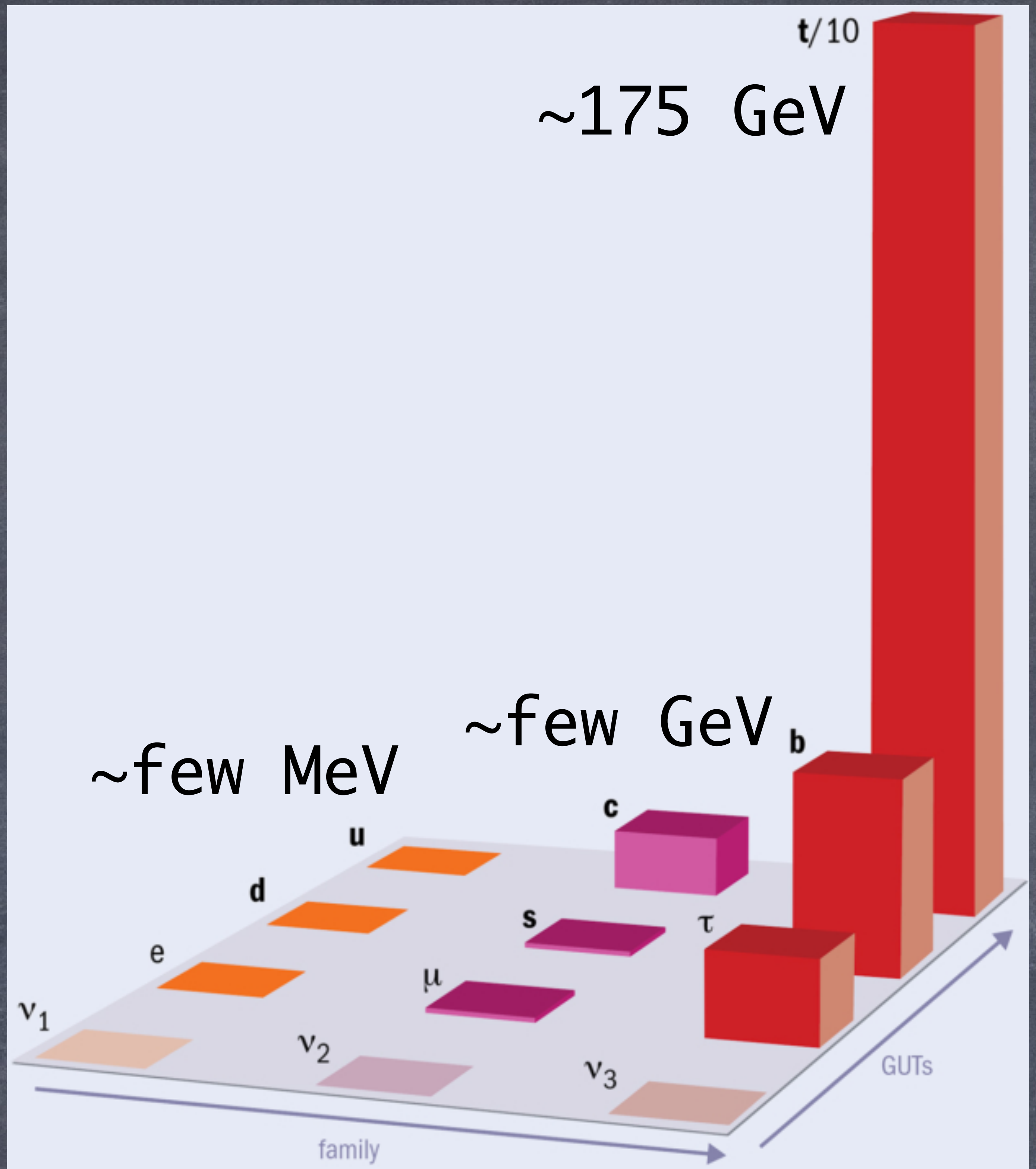
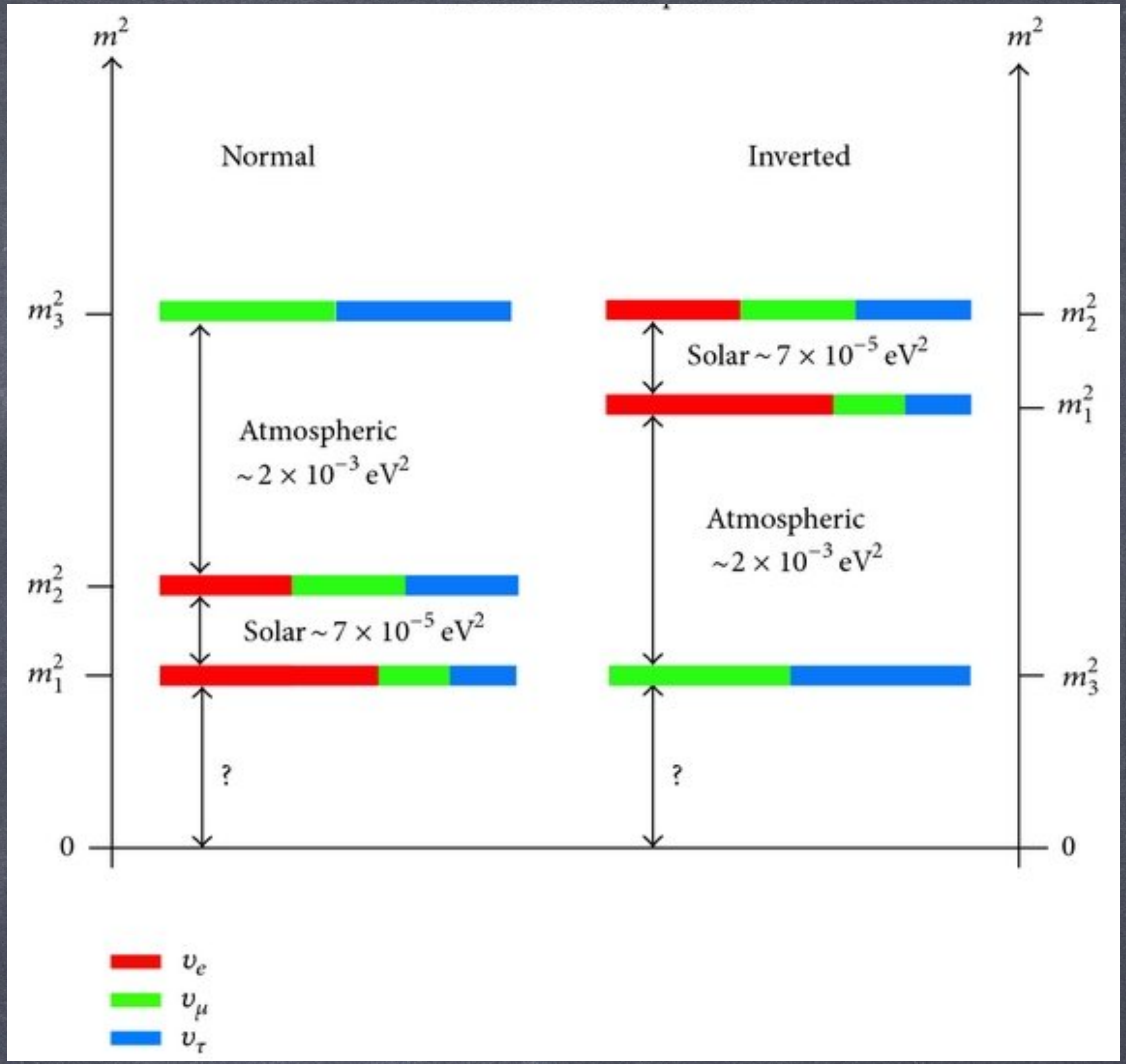
*with Janelly Bautista and Alex Stuart

2312.15391

Motivation

Flavor puzzle

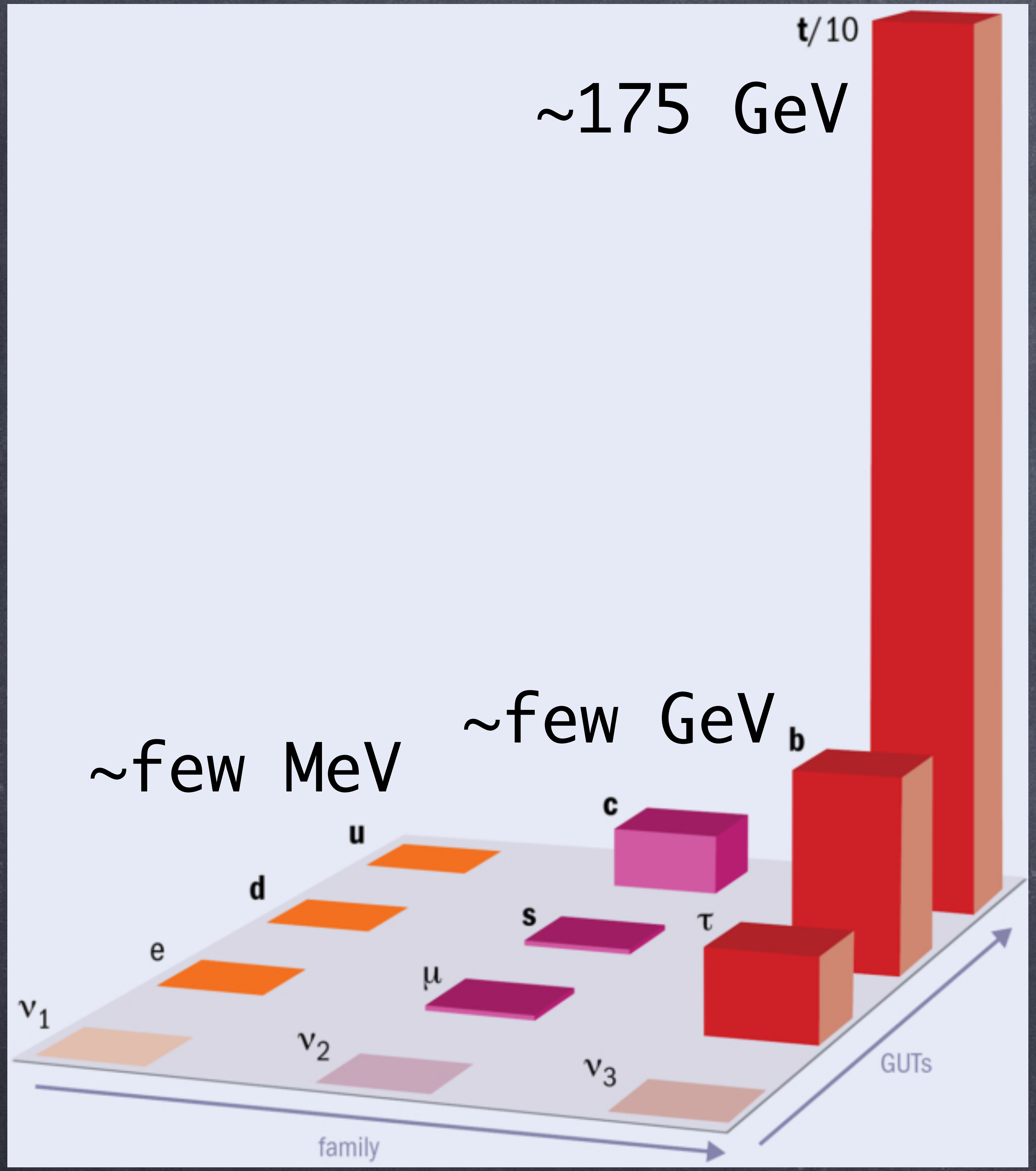


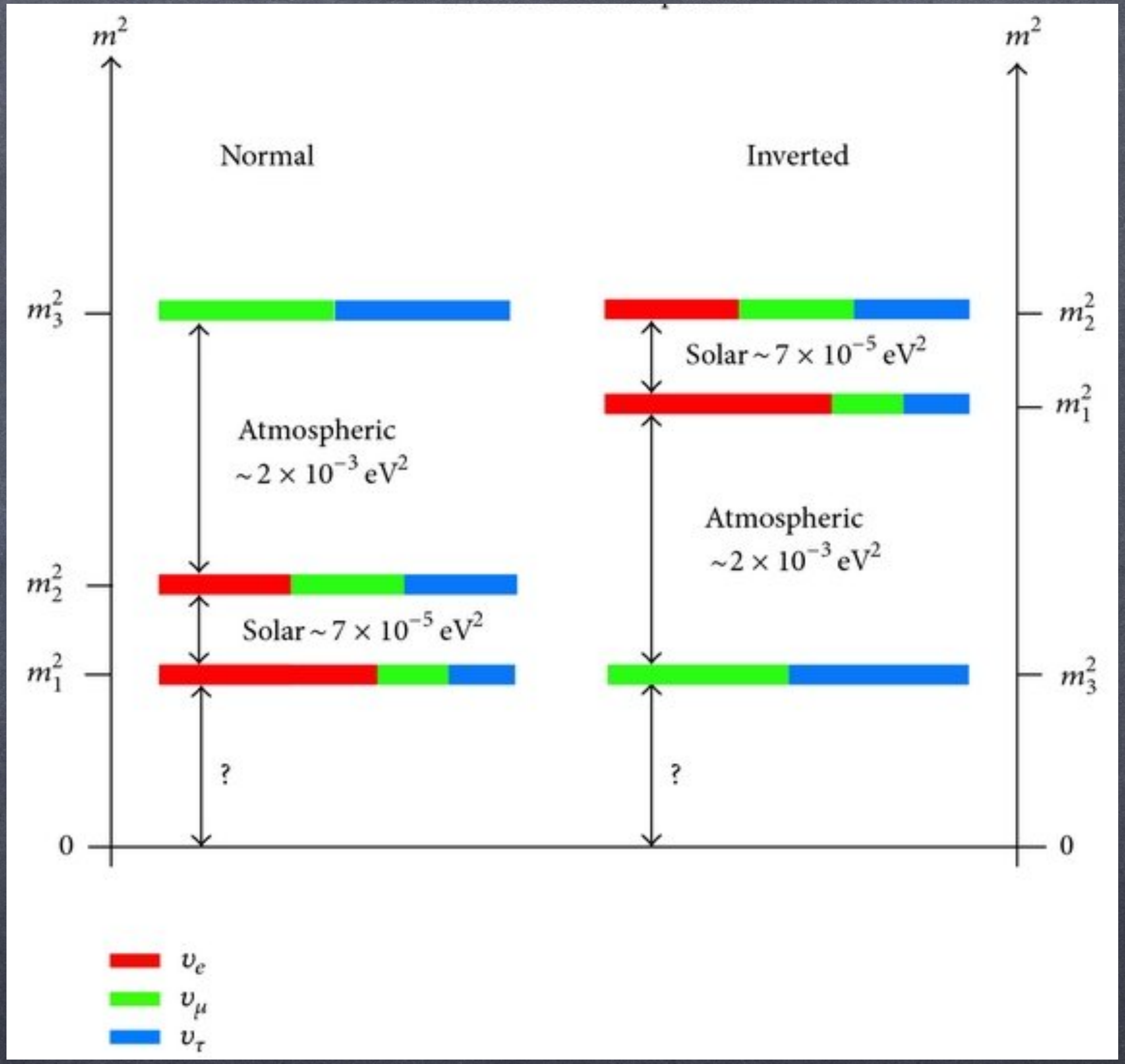


Measurement of the neutrino reactor angle DAYA BAY / RENO / DOUBLE CHOOZ

Ongoing search to determine Dirac-type leptonic CP-violating phase T2K / NOVA

The flavor puzzle





Motivation

The Maki-Nakawa-Sakata-Pontecorvo Matrix

$$-\mathcal{L}_{SM} \supset \frac{g}{\sqrt{2}} \sum_{\ell=e,\mu,\tau} \overline{\ell^-} \gamma^\mu P_L \nu W_\mu^- + \text{H.c.}$$

$$U_{\text{MNSP}} \equiv U_e^\dagger U_\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$\begin{bmatrix} 0.803 \sim 0.845 & 0.514 \sim 0.578 & 0.142 \sim 0.155 \\ 0.233 \sim 0.505 & 0.460 \sim 0.693 & 0.630 \sim 0.779 \\ 0.262 \sim 0.525 & 0.473 \sim 0.702 & 0.610 \sim 0.762 \end{bmatrix}$$

The Particle Data Group (PDG) parametrization

$$U_{\text{MNSP}} = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta_{\text{CP}}} \\ -s_{12}\, c_{23} - c_{12}\, s_{13}\, s_{23}\, e^{i\delta_{\text{CP}}} & c_{12}\, c_{23} - s_{12}\, s_{13}\, s_{23}\, e^{i\delta_{\text{CP}}} & c_{13}\, s_{23} \\ s_{12}\, s_{23} - c_{12}\, s_{13}\, c_{23}\, e^{i\delta_{\text{CP}}} & -c_{12}\, s_{23} - s_{12}\, s_{13}\, c_{23}\, e^{i\delta_{\text{CP}}} & c_{13}\, c_{23} \end{pmatrix}$$

n=3 neutrinos:

3 mixing angles $\theta_{ij} \in [0, \pi/2]$

1 complex phase $\delta \in [0, 2\pi]$

NuFIT 5.2 (2022)

with SK atmospheric data		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
	$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
	$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
	$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
	$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
	$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
	$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
	$\delta_{CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

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so tiny! (2012)

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"solar"

while it may be 0,
fit consistent with
$\approx |\pi/2|$

Motivation

$$Z_2 \times Z_2$$

The Framework

The Klein Symmetry

Everett & Stuart (1501.044336)

Based in the bottom-up approach for residual Klein symmetries

What is this about?

- Assumption I: neutrinos are Majorana

$$M_\nu^T = M_\nu \qquad \text{complex symmetric}$$

- Assumption II: a flavor group at high energy breaks down to a residual symmetry K at low energies

$$G_{\text{flavor}} \longrightarrow K$$

Diagonalization

$$U_\nu^T M_\nu U_\nu = M_\nu^{\text{Diag}} = \text{Diag}(m_1, m_2, m_3) = \text{Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3})$$

$$U_e^\dagger M_e U_e = M_e^{\text{Diag}} = \text{Diag}(|m_e|^2, |m_\mu|^2, |m_\tau|^2)$$

$$M_e = m_e m_e^\dagger$$

$$U_{\text{MNSP}} = U_e^\dagger U_\nu$$

The Framework

Diagonalization

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$$M_e = m_e m_e^\dagger$$

$$U_{\rm MNSP} = U_e^\dagger U_\nu = Q_e U_e^\dagger U_\nu Q_\nu$$

include phases in the Q's

$$M_\nu^{\rm Diag} = Q_\nu^T U_\nu^T M_\nu U_\nu Q_\nu$$

$$Q_\nu = 1, -1$$

$$M_e^{\rm Diag} = Q_e^\dagger U_e^\dagger M_e U_e Q_e$$

$$Q_e = e^{i\beta_i}$$

The Framework

Diagonalization

$$U_\nu^T M_\nu U_\nu = M_\nu^{\rm Diag} = {\rm Diag}(m_1, m_2, m_3) = {\rm Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3})$$

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include phases in the Q's

$$M_\nu^{\rm Diag} = Q_\nu^T U_\nu^T M_\nu U_\nu Q_\nu \qquad M_e^{\rm Diag} = Q_e^\dagger U_e^\dagger M_e U_e Q_e$$

Q_ν = 1,-1 $\qquad Q_e = e^{i\beta_i}$

possible invariances of the ν matrix? $\quad M_\nu^{\rm Diag} = Q_\nu^T M_\nu^{\rm Diag} Q_\nu$

The Framework

Diagonalization

$$U_\nu^T M_\nu U_\nu = M_\nu^{\rm Diag} = {\rm Diag}(m_1, m_2, m_3) = {\rm Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3})$$

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include phases in the Q's

$$M_\nu^{\rm Diag} = Q_\nu^T U_\nu^T M_\nu U_\nu Q_\nu \qquad M_e^{\rm Diag} = Q_e^\dagger U_e^\dagger M_e U_e Q_e$$

Q_ν = 1,-1 $\qquad Q_e = e^{i\beta_i}$

possible invariances of the ν matrix? $\quad M_\nu^{\rm Diag} = Q_\nu^T M_\nu^{\rm Diag} Q_\nu$

$$G_0^{\rm Diag} = 1_{3\times3} \quad G_1^{\rm Diag} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; G_2^{\rm Diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; G_3^{\rm Diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(G_i^{\rm Diag})^2 = 1$, for $i = 0, 1, 2, 3$,
$G_0^{\rm Diag} G_i^{\rm Diag} = G_i^{\rm Diag}$, for $i =$1, 2, 3,
$G_i^{\rm Diag} G_j^{\rm Diag} = G_k^{\rm Diag}$, for $i \neq j \neq k \neq 0$

Klein group $\qquad K \simeq Z_2 \times Z_2$

The Framework

also an invariance of the undiagonalized matrix

$M_\nu = G_i^T M_\nu G_i$ → $G_i = U_\nu G_i^{\rm Diag} U_\nu^\dagger$ $Z_2 \times Z_2$ group again!

After an Euler-angle parametrization of U_ν

$$G_1 = \begin{pmatrix} (G_1)_{11} & (G_1)_{12} & (G_1)_{13} \\ (G_1)^*_{12} & (G_1)_{22} & (G_1)_{23} \\ (G_1)^*_{13} & (G_1)^*_{23} & (G_1)_{33} \end{pmatrix}, G_2 = \begin{pmatrix} (G_2)_{11} & (G_2)_{12} & (G_2)_{13} \\ (G_2)^*_{12} & (G_2)_{22} & (G_2)_{23} \\ (G_2)^*_{13} & (G_2)^*_{23} & (G_2)_{33} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} -c'_{13} & e^{-i\delta} s_{23} s'_{13} & -e^{-i\delta} c_{23} s'_{13} \\ e^{i\delta} s_{23} s'_{13} & s^2_{23} c'_{13} - c^2_{23} & -c^2_{13} s'_{23} \\ -e^{i\delta} c_{23} s'_{13} & -c^2_{13} s'_{23} & c^2_{23} c'_{13} - s^2_{23} \end{pmatrix},$$

the $Z_2 \times Z_2$ group for each low-energy input θ_{ij}, δ !

In addition you get $(M_\nu)_{mn}(\theta_{ij}, \delta)$

Tribimaximal mixing

- Adequate prior to 2010's since $\theta_{13} = 0$

- One column maximally-mixed across THREE entries ("tri")

- One column maximally-mixed across TWO entries ("bi")

- Disfavored after 2010's measurements

$$U^{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Extensively studied under A_4

$$S^2 = T_3^3 = (ST_3)^3 = 1$$

Harrison et al (2002)

He et al (2003)

Xing et al (2006)

etc

Bitrimaximal mixing

- Predicts $\theta_{13} \neq 0$. Favored after Daya Bay

- ONE row and ONE column both maximally-mixed across THREE entries

$$U^{\mathrm{BTM}} = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{3}\right) & 1 & \frac{1}{2}\left(\sqrt{3}-1\right) \\ -1 & 1 & 1 \\ \frac{1}{2}\left(1-\sqrt{3}\right) & 1 & \frac{1}{2}\left(-1-\sqrt{3}\right) \end{pmatrix}$$

Under $\Delta(96)$, $\Delta(96) \times SU(5)$

Toorop et al (2011)

G. J. Ding (2012)

King et al (2013)

etc

The Framework

A common feature

$$U^{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U^{\mathrm{BTM}} = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{3}\right) & 1 & \frac{1}{2}\left(\sqrt{3}-1\right) \\ -1 & 1 & 1 \\ \frac{1}{2}\left(1-\sqrt{3}\right) & 1 & \frac{1}{2}\left(-1-\sqrt{3}\right) \end{pmatrix}$$

Look for vector $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ associated with +1 eigenvalue under $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

$$v_1 = v_2 = v_3$$

U_ν trimaximality $\Longleftrightarrow$ M_ν invariance under $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

a.k.a S

Enforce, say, $G_2(\theta_{ij})$ being equal to S

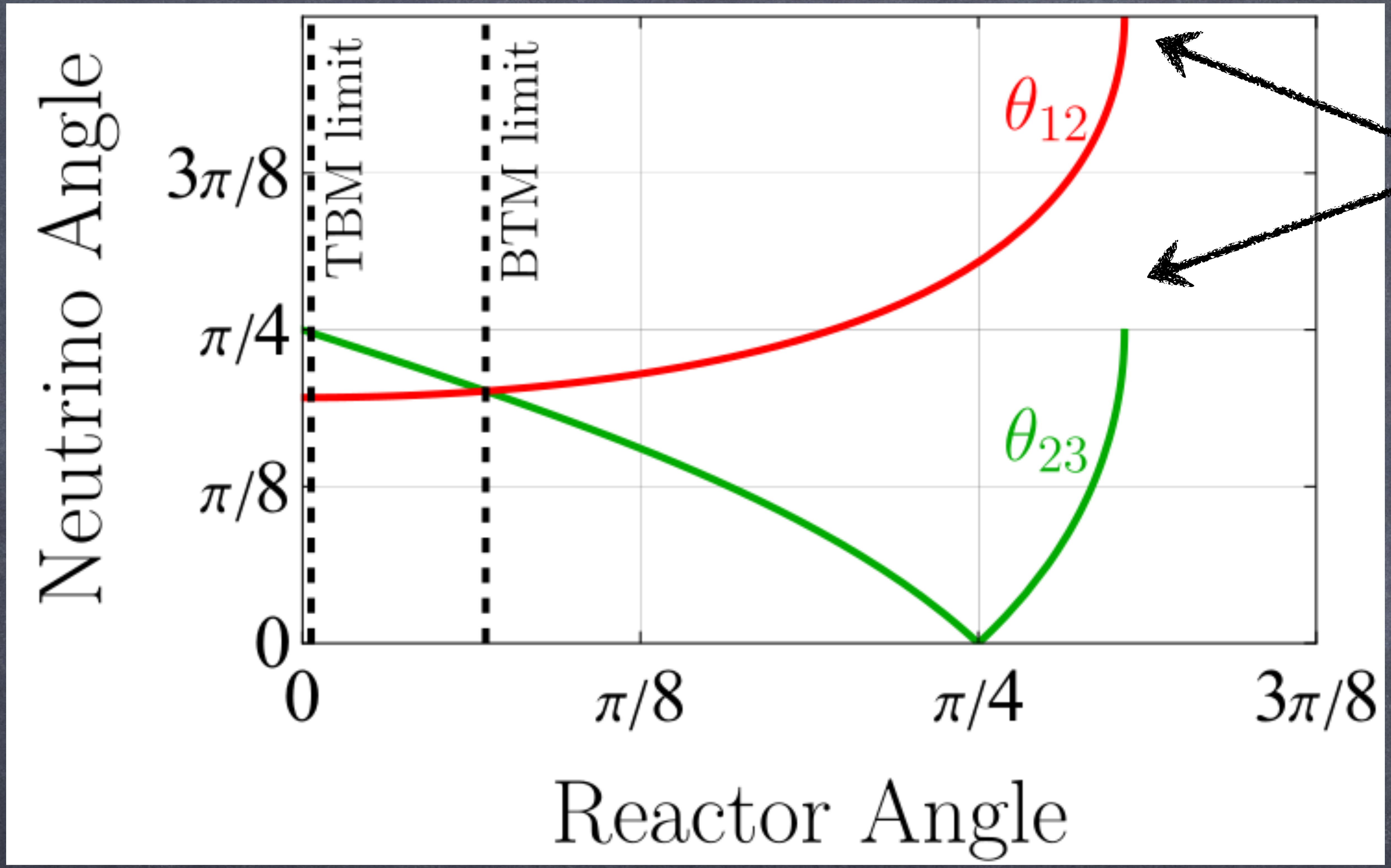
$$\cos(\theta_{23}) = \frac{1}{2}\sec(\theta_{13})\left(\sin(\theta_{13}) + \sqrt{3\cos^2(\theta_{13}) - 1}\right)$$

$$\cos(\theta_{12}) = \frac{\sec(\theta_{13})\sqrt{3\cos(2\theta_{13}) + 1}}{\sqrt{6}}$$

Trimaximality guaranteed for any θ_{13}

The Framework

Trimaximality guaranteed for any θ_{13}



angles turn complex

The Framework

$$\mathrm{TBM} \Rightarrow \mathrm{BTM}$$

The Connection Between Mixings

What are the implications of a bitrimaximal MNSP made up of a tribimaximal neutrino transformation?

this BTM

this TBM

$$U_{\text{MNSP}} = U_e^\dagger U_\nu$$

this $\neq 1$

The U_e matrix

$$U_e = \frac{1}{6}\begin{pmatrix} 2+\sqrt{2}+\sqrt{6} & 2-2\sqrt{2} & 2+\sqrt{2}-\sqrt{6} \\ 2+\sqrt{2}-\sqrt{6} & 2+\sqrt{2}+\sqrt{6} & 2-2\sqrt{2} \\ 2-2\sqrt{2} & 2+\sqrt{2}-\sqrt{6} & 2+\sqrt{2}+\sqrt{6} \end{pmatrix}$$

Entries in a row/column all add up to 1

$$\sum_{i=1}^{3}(U_e)_{ik} = 1,\ \ k = 1,2,3$$

$$\sum_{j=1}^{3}(U_e)_{kj} = 1,\ \ k = 1,2,3$$

In terms of three constants,

$$U_e = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$$

$$a + b + c = 1$$

$$a \equiv \frac{1}{6}(2+\sqrt{2}+\sqrt{6}) \quad \approx 1$$

$$b \equiv \frac{1}{6}(2+\sqrt{2}-\sqrt{6})$$

$$c \equiv \frac{1}{6}(2-2\sqrt{2})$$

Cabbibo-sized (≈ 0.23)

The Connection

Charged lepton mass matrix

$$M_e^{\text{Diag}} = U_e^\dagger M_e U_e = \text{Diag}\{|m_e|^2, |m_\mu|^2, |m_\tau|^2\}$$

M_e won't be so simple. Solving for it,

$$M_e = \begin{pmatrix} a^2|m_e|^2 + b^2|m_\tau|^2 + c^2|m_\mu|^2 & ab|m_e|^2 + ac|m_\mu|^2 + bc|m_\tau|^2 & ab|m_\tau|^2 + ac|m_e|^2 + bc|m_\mu|^2 \\ ab|m_e|^2 + ac|m_\mu|^2 + bc|m_\tau|^2 & a^2|m_\mu|^2 + b^2|m_e|^2 + c^2|m_\tau|^2 & ab|m_\mu|^2 + ac|m_\tau|^2 + bc|m_e|^2 \\ ab|m_\tau|^2 + ac|m_e|^2 + bc|m_\mu|^2 & ab|m_\mu|^2 + ac|m_\tau|^2 + bc|m_e|^2 & a^2|m_\tau|^2 + b^2|m_\mu|^2 + c^2|m_e|^2 \end{pmatrix}$$

$$a \equiv \frac{1}{6}(2 + \sqrt{2} + \sqrt{6}) \quad \approx 1$$

$$b \equiv \frac{1}{6}(2 + \sqrt{2} - \sqrt{6})$$

Cabbibo-sized (≈ 0.23)

$$c \equiv \frac{1}{6}(2 - 2\sqrt{2})$$

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In this parametrization,

$$a \equiv \frac{1}{6}(2+\sqrt{2}+\sqrt{6}) \quad \approx 1$$

$$b \equiv \frac{1}{6}(2+\sqrt{2}-\sqrt{6})$$

Cabbibo-sized (≈ 0.23)

$$c \equiv \frac{1}{6}(2-2\sqrt{2})$$

$$U_e = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$$

$$M_e \approx |m_\tau|^2 \begin{pmatrix} b^2 & bc & ab \\ bc & c^2 & ac \\ ab & ac & a^2 \end{pmatrix}$$

The Connection

Flavor symmetry group

neutrino sector: $Z_2 \times Z_2$

generators

$$U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

undiagonalized G_i elements, $G_i = U_\nu G_i^{\rm Diag} U_\nu^\dagger$ @ TBM

Everett & Stuart (1501.044336)

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Everett & Stuart (1501.044336)

charged leptons: Z_N

$N = 3$

generator

$$T = U_e T^{\rm Diag} U_e^\dagger$$

undiagonalized $T^{\rm Diag} = {\rm Diag}(\omega^2, 1, \omega)$

$\omega = e^{2\pi/3}$

Properties of T

$$T^3 = 1 \qquad T^\dagger M_e T = M_e \qquad (T^2)^\dagger M_e (T^2) = M_e$$

We have checked that

$$(SU)^2 = 1 \qquad (UT^{-1}UT)^3 = 1 \qquad (ST)^3 = 1$$

Crucially, only our chosen T permutation satisfies

$$(UT)^8 = 1 \qquad S = U(UT)^4 U(UT)^4$$

Properties of T

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$$(UT)^8 = 1 \qquad S = U(UT)^4 U(UT)^4$$

2-element presentation rules of $\Delta(96)$

Flavor symmetry group

brute-force scan

96 distinct products of S, T, U

```
{U.T.T.U.T.U, S.T.T.S.U.S.T, S.T.U.S.U.T.S, T.T.U.T.U.T.U, T.U.S.U.S.T.T,
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traces match $\Delta(96)$ character table

$\Delta(96)$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\widetilde{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{3'}$	$\widetilde{\mathbf{3}}'$	$\overline{\mathbf{3}}'$	$\mathbf{6}$
$\mathcal{I}$	1	1	2	3	3	3	3	3	3	6
$3C_4$	1	1	2	$-1+2i$	-1	$-1-2i$	$-1+2i$	-1	$-1-2i$	2
$3C_2$	1	1	2	-1	3	-1	-1	3	-1	-2
$3C'_4$	1	1	2	$-1-2i$	-1	$-1+2i$	$-1-2i$	-1	$-1+2i$	2
$6C''_4$	1	1	2	1	-1	1	1	-1	1	-2
$32C_3$	1	1	-1	0	0	0	0	0	0	0
$12C'_2$	1	-1	0	-1	-1	-1	1	1	1	0
$12C_8$	1	-1	0	i	1	$-i$	$-i$	-1	i	0
$12C'''_4$	1	-1	0	1	-1	1	-1	1	-1	0
$12C'_8$	1	-1	0	$-i$	1	i	i	-1	$-i$	0

$$e^{i\delta}$$

CP violation (?)

So far, our MNSP matrix is bitrimaximal

$$U_{\mathrm{MNSP}}(\theta_{12}^{\mathrm{BTM}}, \theta_{23}^{\mathrm{BTM}}, \theta_{13}^{\mathrm{BTM}}, 0) = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{3}\right) & 1 & \frac{1}{2}\left(\sqrt{3}-1\right) \\ -1 & 1 & 1 \\ \frac{1}{2}\left(1-\sqrt{3}\right) & 1 & \frac{1}{2}\left(-1-\sqrt{3}\right) \end{pmatrix}$$

$$\theta_{12}^{\mathrm{BTM}} = \theta_{23}^{\mathrm{BTM}} = \tan^{-1}\left(\sqrt{3}-1\right) \approx 36.21^\circ$$

$$\theta_{13}^{\mathrm{BTM}} = \sin^{-1}\left(\frac{1}{6}\left(3-\sqrt{3}\right)\right) \approx 12.20^\circ$$

what if...

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$$U_{\text{MNSP}}(\theta_{12}^{\text{BTM}}, \theta_{23}^{\text{BTM}}, \theta_{13}^{\text{BTM}}, \delta) = \begin{pmatrix} \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(3-\sqrt{3}\right)e^{-i\delta} \\ \frac{3\left(\sqrt{3}-1\right)+\left(2\sqrt{3}-3\right)e^{i\delta}}{6\sqrt{3}-15} & \frac{-3+\left(9-5\sqrt{3}\right)e^{i\delta}}{6\sqrt{3}-15} & \frac{1}{\sqrt{3}} \\ \frac{1}{78}\left(12\left(\sqrt{3}-4\right)+\left(9+\sqrt{3}\right)e^{i\delta}\right) & \frac{\left(3-2\sqrt{3}\right)e^{i\delta}+3-3\sqrt{3}}{6\sqrt{3}-15} & -\frac{1}{6}\left(3+\sqrt{3}\right) \end{pmatrix}$$

we'll keep the name, $U^{\text{BTM}}(\delta)$

CP violation

What happens with our earlier BTM-TBM connection? Earlier,

this BTM → ; this TBM →

$$U_{\text{MNSP}} = U_e^\dagger U_\nu$$

this $\neq 1$

Now,

$$U_{\text{MNSP}}(\delta) = U_e^\dagger(\delta) U_\nu(\delta)$$

CP violation

What happens with our earlier BTM-TBM connection? Earlier,

this BTM → U_{MNSP}, this TBM → U_ν, this $\neq 1$ → $U_e^\dagger$

$$U_{\mathrm{MNSP}} = U_e^\dagger U_\nu$$

Now,

[erase TBM argument]

$$U_{\mathrm{MNSP}}(\delta) = U_e^\dagger(\delta) U_\nu(\delta)$$

The δ-dependence can only come from $U_e(\delta)$. This matrix isn't friendly though,

$$U_e(\delta)_{13} = \frac{1}{36-30\sqrt{3}}\left(-6(-1-4\sqrt{2}+\sqrt{3}+2\sqrt{6}) + (6-3\sqrt{2}-4\sqrt{3}+\sqrt{6})e^{-i\delta}\right)$$

etc.

Only U_e acquires the CPV phase. This affects the generator in the charged lepton sector

$$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$T = U_e T^{\rm Diag} U_e^\dagger \longrightarrow T(\delta) = U_e(\delta) T^{\rm Diag} U_e(\delta)^\dagger$$

CP violation

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The neutrino sector generators still defined through the old $U^{\rm TBM}$

$$U = U^{\rm TBM} G_3^{\rm Diag} U^{\rm TBM\dagger}$$

$$S = U^{\rm TBM} G_2^{\rm Diag} U^{\rm TBM\dagger}$$

With some luck, the $T(\delta)$, U, and S satisfy algebraic relations

CP violation

$T(\delta)$ is an order-3 element,

$$T(\delta)^3 = 1 \quad ✓$$

CP violation

$T(\delta)$ is an order-3 element,

$$T(\delta)^3 = 1 \quad ✓$$

There's more, with U it satisfies

$$[UT^{-1}(\delta)UT(\delta)]^3 = 1 \quad ✓ \qquad\qquad [UT(\delta)]^8 = 1 \quad ✓$$

Not expected at all! While U is made of 0's and -1's, $T(\delta)$ is highly nontrivial (square roots and δ's everywhere)

Be cautious though,

$$(SU)^2 = 1 \quad ✓ \qquad\qquad [ST(\delta)]^3 \neq 1 \quad ✗$$

The usual group element S doesn't do the job here. However, in the 2-element presentation of $\Delta(96)$ the S was never independent,

$$S = U(UT)^4U(UT)^4$$

CP violation

Be cautious though,

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The usual group element S doesn't do the job here. However, in the 2-element presentation of $\Delta(96)$ the S was never independent,

$$S = U(UT)^4U(UT)^4$$

Perhaps the δ-dependent version of it works?

Define

$$\mathcal{S} = U[UT(\delta)]^4 U[UT(\delta)]^4$$

Horrendous as a matrix product, yet

$$\mathcal{S}^2(\delta) = 1 \quad ✓$$

$$[\mathcal{S}(\delta)U]^2 = 1 \quad ✓$$

Define

$$\mathcal{S} = U[UT(\delta)]^4 U[UT(\delta)]^4$$

Horrendous as a matrix product, yet

$$\mathcal{S}^2(\delta) = 1 \quad ✓$$

$$[\mathcal{S}(\delta)U]^2 = 1 \quad ✓$$

and as a bonus

$$\left[\mathcal{S}(\delta)T(\delta)\right]^3 = 1 \quad ✓✓$$

The group elements U, $\mathcal{S}(\delta)$, and $T(\delta)$ fulfill the $\Delta(96)$ symmetry!

BIG question

Is this δ physical?

CP violation

The group elements U, $\mathcal{S}(\delta)$, and $T(\delta)$ fulfill the $\Delta(96)$ symmetry!

BIG question

Is this δ physical?

NO

Every single algebraic relation holds for any value of δ

Concluding remarks

- We exploited the trimaximal feature of two mixing patterns (TBM & BTM) to extract the parametric behavior of two angles in function of a third one
- We briefly looked at the unitary matrix U_e in the charged leptons that, together with a tribimaximal neutrino transformation, results in a bitrimaximal MNSP
- The flavor group in the connection above is found to be $\Delta(96)$ ~~by brute force~~
- The group persists as δ is included. However, this CPV phase fails to be physical

Concluding remarks

- We exploited the trimaximal feature of two mixing patterns (TBM & BTM) to extract the parametric behavior of two angles in function of a third one

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- The flavor group in the connection above is found to be $\Delta(96)$ ~~by brute force~~

- The group persists as δ is included. However, this CPV phase fails to be physical

¡Gracias!

BACKUP

TBM & BTM predictions

$$\theta_{12}^{\rm TBM} = \cos^{-1}\left(\sqrt{2/3}\right) \approx 35.24^\circ, \quad \theta_{13}^{\rm TBM} = 0^\circ, \quad \theta_{23}^{\rm TBM} = \pi/4 = 45^\circ$$

$$\theta_{12}^{\rm BTM} = \theta_{23}^{\rm BTM} = \tan^{-1}\left(\sqrt{3}-1\right) \approx 36.21^\circ, \quad \theta_{13}^{\rm BTM} = \sin^{-1}\left(\frac{1}{6}\left(3-\sqrt{3}\right)\right) \approx 12.20^\circ.$$

BACKUP

T-generator permutations

$$T_1^{\rm Diag} = {\rm Diag}(1,\omega,\omega^2),\ T_2^{\rm Diag} = {\rm Diag}(\omega^2,1,\omega),\ T_3^{\rm Diag} = {\rm Diag}(\omega,\omega^2,1),$$
$$(T_1^{\rm Diag})^2 = {\rm Diag}(1,\omega^2,\omega),\ (T_2^{\rm Diag})^2 = {\rm Diag}(\omega,1,\omega^2),\ (T_3^{\rm Diag})^2 = {\rm Diag}(\omega^2,\omega,1),$$

$$\omega^2 T_1^{\rm Diag} = T_2^{\rm Diag} = \omega T_3^{\rm Diag} \qquad \text{and} \qquad \omega (T_1^{\rm Diag})^2 = (T_2^{\rm Diag})^2 = \omega^2 (T_3^{\rm Diag})^2.$$

BACKUP